

R-CLOSED HOMEOMORPHISMS ON SURFACES

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ABSTRACT. Let f be an R -closed homeomorphism on a connected orientable closed surface M . In this paper, we show that if M has genus more than one, then each minimal set is either a periodic orbit or an extension of a Cantor set. If $M = \mathbb{T}^2$ and f is neither minimal nor periodic, then either each minimal set is a finite disjoint union of essential circloids or there is a minimal set which is an extension of a Cantor set. If $M = \mathbb{S}^2$ and f is not periodic but orientation-preserving (resp. reversing), then the minimal sets of f (resp. f^2) are exactly two fixed points and other circloids and $\mathbb{S}^2/\tilde{f} \cong [0, 1]$.

1. INTRODUCTION

In [Ma], it has shown that if f is orientation-preserving R -closed and non-periodic homeomorphism on \mathbb{S}^2 , then f has exactly two fixed points and every non-degenerate orbit closure is a homology 1-sphere. In this paper, we consider minimal sets of R -closed homeomorphisms on closed surfaces. Precisely, let f be an R -closed homeomorphism on a connected orientable closed surface M . Then we show that if M has genus more than one, then each minimal set is either a periodic orbit or an extension of a Cantor set. If $M = \mathbb{T}^2$ and f is neither minimal nor periodic, then either the orbit class space \mathbb{T}^2/\tilde{f} is a 1-manifold and each minimal set is a finite disjoint union of essential circloids, or there is a minimal set which is an extension of a Cantor set. If $M = \mathbb{S}^2$ and f is not periodic but orientation-preserving (resp. reversing), then the minimal sets of f (resp. f^2) are exactly two fixed points and other circloids and $\mathbb{S}^2/\tilde{f} \cong [0, 1]$. Finally we state the applications for codimension two foliations.

2. PRELIMINARIES

By a flow, we mean a continuous action of a topological group G on a topological space X . We call that G is R -closed if $R := \{(x, y) \mid y \in \overline{G(x)}\}$ is closed. Recall that a subset S of G is said to be (left) syndetic if there is a compact set K of G with $KS = G$. For a point $x \in X$ and an open U of X , let $N(x, U) = \{g \in G \mid gx \in U\}$. We say that x is an almost periodic point if $N(x, U)$ is syndetic for every neighborhood U of x . A flow G is pointwise almost periodic if every point $x \in X$ is almost periodic. When X is a compact metrizable (i.e. compact Hausdorff) space, they are known that if f is R -closed, then f is pointwise almost periodic, and that f is pointwise almost periodic if and only if $\{\overline{G(x)} \mid x \in X\}$ is a decomposition of X . When f is pointwise almost periodic, write $\hat{\mathcal{F}} := \{\overline{G(x)} \mid x \in X\}$ the decomposition of X . Note that $X/\hat{\mathcal{F}}$ is called an orbit class space and is also denoted by X/\tilde{G} . A

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pointwise almost periodic flow G is weakly almost periodic in the sense of Gottschalk [G] if the saturation of orbit closures for any closed subset of X is closed (i.e. the quotient map $X \rightarrow X/\tilde{G}$ is closed). By Theorem 5 [Ma] and Proposition 1.2 [Y], the following are equivalent for a pointwise almost periodic flow f on a compact metrizable space: 1) $\hat{\mathcal{F}}$ is R -closed, 2) $\hat{\mathcal{F}}$ is weakly almost periodic, 3) $\hat{\mathcal{F}}$ is upper semi-continuous, 4) $X/\hat{\mathcal{F}}$ is Hausdorff.

By a continuum we mean a compact connected metrizable space which is not a singleton. A continuum $A \subset X$ is said to be annular if it has a neighbourhood $U \subset X$ homeomorphic to an open annulus such that $U - A$ has exactly two components each of which is homeomorphic to an annulus. We call any such U an annular neighbourhood of A . We say a subset $C \subset X$ is a circloid if it is an annular continuum and does not contain any strictly smaller annular continuum as a subset. For a subset A of X and a decomposition $\hat{\mathcal{F}}$, the saturation $\text{Sat}(A)$ of A is the union $\cup\{L \in \hat{\mathcal{F}} \mid A \cap L \neq \emptyset\}$ of elements of $\hat{\mathcal{F}}$ intersecting A .

Lemma 2.1. *Let X be a sequentially compact space and (C_n) a sequence of connected subsets of X . Suppose that there are disjoint open subsets U, V of X and sequences (x_n) (resp. (y_n)) converging to $x \in U$ (resp. $y \in V$) with $x_n, y_n \in C_n$. Then there is an element $z \in (\cap_{n>0} \overline{\cup_{k>n} C_k}) \setminus U \sqcup V$.*

Proof. Let $F = X - U \sqcup V$ be a closed subset. Since C_n is connected, each C_n intersects F . Choose $z_n \in C_n \cap F$. Since X is sequentially compact, we have F is also sequentially compact. Hence there is a convergent subsequence of z_n and so the limit $z \in F$ is desired. \square

We show that connected closures for an R -closed flow must converge to a connected closure.

Lemma 2.2. *Let G be an R -closed flow on a sequentially compact space X and let (w_n) be a convergent sequence to a point $w \in M$. If each $\overline{G(w_n)}$ is connected, then the closure $\overline{G(w)}$ is connected.*

Proof. Put $C_n := \overline{G(w_n)}$. Suppose that $\overline{G(w)}$ is disconnected. Then there are disjoint open subsets U, W of M such that $\overline{G(w)} \subseteq U \sqcup W$, $\overline{G(w)} \cap U \neq \emptyset$, and $\overline{G(w)} \cap W \neq \emptyset$. Then $G(w) \cap U \neq \emptyset$ and $G(w) \cap W \neq \emptyset$. Since w_n converges to w , the continuity of G implies that there are sequences (x_n) (resp. (y_n)) converging $x \in U$ (resp. $y \in W$) such that $x_n, y_n \in C_n$. By Lemma 2.1, there is an element $z \in (\cap_{n>0} \overline{\cup_{k>n} C_k}) \setminus U \sqcup W$. Then there is a convergent sequence $(z_n \in C_n)$ to z . Since $z_n \in C_n = \overline{G(w_n)}$ and $z \notin \overline{G(w)}$, we obtain $(w_n, z_n) \in R$ and $(w, z) \notin R$. This contradicts the R -closedness. Therefore C is connected. \square

Let f be a pointwise almost periodic homeomorphism on an orientable connected closed surface M . Recall $\hat{\mathcal{F}} = \hat{\mathcal{F}}_f = \{\overline{O_f(x)} \mid x \in M\}$. Write $V = V_f := \{x \in M - \text{Fix}(f) \mid \overline{O(x)} \text{ is connected}\} \cup \{L \in \hat{\mathcal{F}} : \text{connected}\} - \text{Fix}(f)$.

Lemma 2.3. *If f is not minimal, then V consists of circloids.*

Proof. Since f is pointwise almost periodic, we have that the non-wandering set of f is M . By Theorem 1.1.[K], each element C in V of $\hat{\mathcal{F}}$ is annular. Let U be a sufficiently small annular neighbourhood of C such that $U - C$ is a disjoint union of two open annuli A_1, A_2 . Since C is f -invariant and minimal, we have that $C = \partial A_1 \cap \partial A_2$. Suppose that there is an annular continuum $C' \subsetneq C$. Then there

is an annular neighbourhood U' of C' such that $U' \subset U$. Embedding U into \mathbb{S}^2 , we may assume that U is a subset of \mathbb{S}^2 . Then $\mathbb{S}^2 - C$ is a disjoint union of two open disks D_1, D_2 and $\mathbb{S}^2 - C'$ is a disjoint union of two open disks D'_1, D'_2 . Since $\mathbb{S}^2 - C' \supsetneq \mathbb{S}^2 - C$, we have $D_1 \sqcup D_2 \subsetneq D'_1 \sqcup D'_2$. Since $D_1 \sqcup D_2 \sqcup \{x\}$ for any element $x \in C$ is connected, we obtain $D'_1 \sqcup D'_2$ is connected. This contradicts to disconnectivity. Thus C is a circloid. \square

Note that a point x is almost periodic if and only if for every open neighborhood U of x , there is $K \in \mathbb{Z}_{\geq 0}$ such that $\mathbb{Z} = \{n, n+1, \dots, n+K \mid n \in N(x, U)\}$. The above lemmas implies the following statement.

Corollary 2.4. *Suppose f is not minimal but R -closed. Each point of $\overline{V} - V$ is a fixed point.*

Taking the iteration, we obtain the following corollary.

Corollary 2.5. *Suppose f is not minimal. For any $x \in M$, if $\overline{O(x)}$ is not periodic but consists of finitely many connected components, then $\overline{O(x)}$ consists of circloids.*

Proof. Let k be the number of the connected components of $\overline{O(x)}$. By Theorem I [ES], we have f^k is also pointwise almost periodic and so $\overline{O_{f^k}(x)}$ is connected. By Lemma 2.3, $\overline{O_{f^k}(x)}$ is a circloid. Since each connected component of $\overline{O(x)}$ is homeomorphic to each other, the assertion holds. \square

This corollary can sharpen Theorem 6 [Ma] into the following statement.

Corollary 2.6. *Let f be a non-periodic R -closed orientation-preserving (resp. reversing) homeomorphism on \mathbb{S}^2 . Then \mathbb{S}^2/\tilde{f} is a closed interval and $\hat{\mathcal{F}}_f$ (resp. $\hat{\mathcal{F}}_{f^2}$) consists of two fixed points and other circloids.*

Proof. Suppose that f is orientation-preserving. By Theorem 3 and 6 [Ma], there are exactly two fixed points and all other orbit closures of f are connected. By Lemma 2.3, they are circloids. We show that $M/\hat{\mathcal{F}}$ is a closed interval. Indeed, let A be the sphere minus two fixed points. Suppose that there is a circloid L which is null homotopic in A . Let D be a disk bounded by L in A . Since M consists of non-wandering points, the Brouwer's non-wandering Theorem [B] to D implies that $f|_D$ has a fixed point. This contradicts to the non-existence of fixed points in A . On the orientation reversing case, since f^2 is orientation-preserving, the assertion holds. \square

Note that if there is a dense orbit and $\hat{\mathcal{F}}$ is pointwise almost periodic, then $\hat{\mathcal{F}}$ is minimal and $V = T^2 = M$. Now we proof a key lemma.

Lemma 2.7. *Suppose that f is an orientation-preserving (resp. reversing) R -closed homeomorphism on an orientable connected closed surface M . If there is a minimal set which is a circloid, then $M/\hat{\mathcal{F}} = V/\hat{\mathcal{F}}$ is a closed interval or a circle. Moreover either $M \cong \mathbb{S}^2$ and $\hat{\mathcal{F}}_f$ (resp. $\hat{\mathcal{F}}_{f^2}$) consists of exactly two fixed points and other circloids, or $M \cong \mathbb{T}^2$ and $\hat{\mathcal{F}}_f$ (resp. $\hat{\mathcal{F}}_{f^2}$) consists of essential circloids.*

Proof. Fix a metric compatible to the topology of M . First, suppose that f is orientation-preserving. First we show that V is open. Let L be a circloid of $\hat{\mathcal{F}}$ with a sufficiently small annular neighbourhood A . Since $\hat{\mathcal{F}}$ is R -closed, Lemma 1.6 [Y] implies that the quotient map $M \rightarrow M/\hat{\mathcal{F}}$ is closed and so the saturation

$F := \text{Sat}(M - A)$ is closed and nonempty. Since $M/\hat{\mathcal{F}}$ is compact Hausdorff, there is a small number $\varepsilon > 0$ such that F does not intersect with the closure of the ε -neighborhood $B_\varepsilon(L)$ of L . Since $\hat{\mathcal{F}}$ is upper semi-continuous, there is an open saturated neighbourhood $U \subseteq B_\varepsilon(L)$ of L . Since $F \supseteq M - A$ and $F \cap \overline{B_\varepsilon(L)} = \emptyset$, we have $\overline{U} \subseteq A$. Since M is compact metrizable and since $M - U$ and L are disjoint closed, there is some $\varepsilon' \in (0, \varepsilon)$ such that the open ε' -neighborhood $V := B_{\varepsilon'}(L)$ of L contained in U . Then V is arcwise connected. Since L is invariant and contained in V , we have that $L \subset f^k(V)$ for any $k \in \mathbb{Z}$ and so the saturation $\text{Sat}(V) = \cup_{k \in \mathbb{Z}} f^k(V)$ is arcwise connected open and is contained in U . By replacing U by $\text{Sat}(V)$, we may assume that U is arcwise connected. Define $\text{Fill}(U) := \cup \{B \subset A : \text{disk} \mid \partial B = \gamma \text{ for some loop } \gamma \subset U\}$. We show that $\text{Fill}(U)$ is annular. Indeed, since $U \subseteq B_\varepsilon(L)$, we have $\text{Fill}(U) \subseteq B_\varepsilon(L) \subseteq A$. Since A is an open annulus, we can consider an embedding of A to a sphere S such that the complement of L consists of two open disks D_1, D_2 . Then $A \cup D_i$ is an open disk. For each connected component B of $A - U$ which is contained by a disk bounded by a loop in U , the saturation $\text{Sat}(B)$ is contained in a union of disks bounded by loops in U . Therefore $\text{Fill}(U) \cup D_i = \cup \{B \subset A \cup D_i : \text{disk} \mid \partial B = \gamma \text{ for some loop } \gamma \subset U \cup D_i\}$ is a simply connected open subset of the disk $A \cup D_i$. By Riemann mapping theorem, this implies that $\text{Fill}(U) \cup D_i$ is homeomorphic to an open disk and so $\text{Fill}(U)$ is annular. Note that $\text{Fill}(U)$ is saturated. Let N be the two points compactification of $\text{Fill}(U)$ and f' the resulting homeomorphism on N adding new two fixed points. Then N is a sphere and L is also a minimal set of f' which separates the new two fixed points. Since M/\tilde{f} is normal, the closedness of $M - \text{Fill}(U)$ implies that N/\tilde{f}' is Hausdorff. Hence we have f' is R -closed. By Corollary 2.6, the orbit closures on N are exactly two fixed points and other circloids. This implies that V is open and the quotient of each connected component of V is totally ordered. By Corollary 2.4, each boundary of V is a fixed point and so M/\mathcal{F} is an closed interval or a circle. If V has nonempty boundaries, then M is a sphere and if V has no boundaries then M is a torus. This completes a proof of the orientable case. Suppose that f is orientation-reversing. Since f^2 is orientation-preserving and since $M/\hat{\mathcal{F}}_{f^2}$ is a double branched covering of $M/\hat{\mathcal{F}}$, we have that $M/\hat{\mathcal{F}}$ is a closed interval. \square

In the higher genus case, we obtain the following corollary.

Corollary 2.8. *Let f be a R -closed homeomorphism on a closed surface with genus more than one. Then each non-periodic minimal set of f has infinitely many connected components.*

Proof. Suppose that there is a non-periodic minimal set \mathcal{M} of f with at most finitely many connected components. Let k be the number of connected components of \mathcal{M} . Then each connected component \mathcal{M}' of \mathcal{M} is a minimal set of f^k . By Lemma 2.3, we obtain that \mathcal{M}' is a circloid. By Corollary 2.5, we have M is \mathbb{S}^2 or \mathbb{T}^2 . This contradicts to the hypothesis. \square

3. MAIN RESULTS AND THEIR PROOFS

We say that a minimal set \mathcal{M} on a surface homeomorphism $f : S \rightarrow S$ is an extension of a Cantor set (resp. a periodic orbit) if there are a surface homeomorphism $\tilde{f} : S \rightarrow S$ and a surjective continuous map $p : S^2 \rightarrow S^2$ which is homotopic

to the identity such that $p \circ f = \tilde{f} \circ p$ and $p(\mathcal{M})$ is a Cantor set (resp. a periodic orbit) which is a minimal set of \tilde{f} . Now we state main results.

Theorem 3.1. *Let M be a connected orientable closed surface with genus more than one. Each minimal set of an R -closed homeomorphism on M is either a periodic orbit or an extension of a Cantor set.*

Proof. Let \mathcal{M} be a minimal set. By Lemma 2.7, M is not a finite disjoint union of circloids. By Theorem [PX], we have that \mathcal{M} is an extension of either a periodic orbit or a Cantor set. We may assume that \mathcal{M} is an extension of a periodic orbit. By the proof of Addendum 3.17 [JKP] and Proposition 5.1 [PX], we obtain that \mathcal{M} has at most finitely many connected components. By Corollary 2.8, this minimal set \mathcal{M} is a periodic orbit. \square

In the toral case, we obtain the following statement.

Theorem 3.2. *Each R -closed toral homeomorphism f satisfies one of the following:*

1. *f is minimal.*
2. *f is periodic.*
3. *Each minimal set is finite disjoint union of essential circloids.*
4. *There is a minimal set which is an extension of a Cantor set.*

Proof. Suppose that f is neither minimal nor periodic and there are no minimal sets which are extensions of Cantor sets. Since f is not periodic, by Theorem 4 [JKP], there is a minimal set \mathcal{M} which is a finite disjoint union of circloids. Let k be the number of the connected components of \mathcal{M} . By Theorem 1.1 [Y2], the iteration f^k is also R -closed. Applying Lemma 2.7 to f^k , we have that each minimal set of f is a finite disjoint union of essential circloids. \square

Recall that f is aperiodic if f has no periodic orbits. By Theorem D [J], we obtain the following corollary.

Corollary 3.3. *Each orbits closure of a non-minimal aperiodic R -closed toral homeomorphism isotopic to identity is a circloid.*

4. APPLICATIONS TO CODIMENSION TWO FOLIATIONS

In [Y], it show that a foliated space on a compact metrizable space which is minimal or “compact and without infinite holonomy”, is R -closed. Since each compact codimension two foliation on a compact manifold has finite holonomy [Ep] [V], we have that the set of minimal or compact codimension two foliations is contained in the set of codimension two R -closed foliations. The following examples are codimension two R -closed foliations which are neither minimal nor compact. Considering an axisymmetric embedding of \mathbb{T}^2 (resp. \mathbb{S}^2) into \mathbb{R}^3 , any irrational rotation on it around the axis is a non-periodic R -closed homeomorphism. Taking a suspension on \mathbb{T}^2 (resp. \mathbb{S}^2), we obtain the following statement by Theorem 3.2 (resp. Corollary 2.6).

Corollary 4.1. *Each suspension of a R -closed homeomorphism on \mathbb{T}^2 or \mathbb{S}^2 which is neither minimal nor periodic induces a codimension two R -closed foliation which is neither minimal nor compact. Moreover there are such homeomorphisms on \mathbb{T}^2 and \mathbb{S}^2 .*

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